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# TRANSVERSE CYLINDRICAL SIMPLE WAVES AND SHOCK WAVES IN ELASTIC NON-CONDUCTORS

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**Abstract**—A modulated simple wave theory is developed for transverse cylindrical motions of an unstrained incompressible isotropic elastic non-conductor with the aid of a modified version of Hunter and Keller's "Weakly Nonlinear Geometrical Optics" method. This theory is then used to construct shock wave solutions using the shock-fitting method. The evolution law thus derived shows that the effect of nonlinearity on the evolution of transverse cylindrical shock waves is cumulative, but that by the time it becomes most pronounced, geometrical spreading has already attenuated the shock amplitude until it is exponentially small. It follows that the linear theory gives satisfactory results for the propagation of transverse cylindrical shock waves. This is in sharp contrast to the situation for plane transverse shock waves whose amplitudes decay in the presence of material nonlinearities whilst the linear theory predicts constant amplitudes. Where it is present, geometrical spreading would appear to be a more potent decay mechanism than material non-linearity.

## 1. INTRODUCTION

The most important property of the effects of weak nonlinearity on wave propagation is that they are cumulative and are significant only for great enough distances of travel. It is therefore of interest to investigate how the evolution of shock waves is affected by material nonlinearity, geometrical spreading and internal dissipation, and especially, how important the former is in the presence of the last two so that we can obtain a clear understanding of the validity of linear theories. In a previous paper (see Fu and Scott, 1989c), we have studied the combined effects of material nonlinearity and geometrical spreading on the propagation of dilatational spherical and cylindrical shock waves. In this paper, it is the propagation of transverse cylindrical shock waves which concerns us.

Shock waves are very different in their evolutionary behaviour from acceleration waves or other higher order discontinuities and are much more difficult to analyze. The difficulty lies in that shock waves are always coupled in their evolutionary behaviour with the other higher order discontinuities which accompany them. If one follows the same procedure as for acceleration waves, one finds that the shock velocity depends on the shock amplitude, whilst the shock amplitude is governed by an evolution equation which also involves the amplitude of the accompanying second-order discontinuity. We derive a second evolution equation for the amplitude of the accompanying second-order discontinuity which involves the amplitude of the accompanying third-order discontinuity. This procedure could be carried out to higher orders. In the case of curved shock waves, the evolution of shock waves is further complicated by geometrical considerations, for, in general, the shock surface geometry has to be determined together with the shock amplitude. It is in general, therefore, not possible to derive an exact evolution law. It is possibly because of this that works on the global evolutionary behaviour of shock waves are relatively sparse in comparison with the abundant literature on acceleration waves; previous studies have been in the main confined to the general properties and the instantaneous decay behaviour of shock waves, see, for example, Eringen and Suhubi (1974), McCarthy (1975), Wesolowski and Burger (1977) and Ukeje (1981, 1982). The exact evolution equations for a curved shock wave of the most general geometry and its accompanying second-order discontinuity have been neatly derived by Li and Ting (1982) and Ting and Li (1983), who have also suggested several choices of the direction in which the growth or decay of the discontinuity

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is measured in order to simplify the evolution equations. However, they do not discuss solutions of these equations.

Fu and Scott have employed two different approximate methods to derive asymptotic evolution laws for each of plane dilatational shock waves (1989a), plane transverse shock waves (1989b) and curved dilatational shock waves (1989c). The first approximate method, which they call the singular surface method, is a combination of singular surface theory and perturbation methods. The second method is the shock-fitting method based on simple wave theories. The two methods are complementary to each other and have differing advantages and disadvantages. The singular surface method is well-adapted to all problems of shock wave propagation, but it needs to be underpinned by an assumption concerning the amplitude of the accompanying third-order discontinuity which is such as to allow the term containing this quantity to be neglected. The shock-fitting method, on the other hand, gives more complete results than the singular surface method, but since it is based upon simple wave theories (and exact simple wave solutions exist only in plane motions of elastic non-conductors for which the isentropic assumption is valid), the shock-fitting method can only be applied to curved shock waves or shock waves travelling in materials with internal dissipation when an approximate simple wave solution can be found. In Fu and Scott's (1989c) discussion of curved dilatational shock waves, a modulated simple wave theory, which is valid under the small-amplitude, finite-rate assumption, was established with the aid of Hunter and Keller's (1983) "Weakly Nonlinear Geometrical Optics" (hereinafter WNGO) method.

The specific problem we consider in this paper is that of an initially unstrained incompressible isotropic elastic non-conductor with a circular cylindrical cavity of radius $r_0$ which is set in motion by prescribing the azimuthal velocity at the internal boundary $r = r_0$ for all positive times. By integration of the prescribed velocity at the boundary we see that this problem is equivalent to a displacement boundary value problem. The material is taken to be incompressible to ensure that transverse waves may propagate; it is taken to be unstrained and isotropic to ensure that the initially circular wave fronts remain so as they propagate through the material.

It is because of the more general material constitution of solids that the transverse motions discussed here are possible, even though they cannot occur in gases. This makes the present problem of special interest since it can be shown that it cannot be treated by Varley and Cumberbatch's (1966) theory of relatively undistorted waves or by Hunter and Keller's WNGO method and neither can it be treated by Anile's (1984) Generalized Wavefront Expansion method. All of these methods have been shown to give satisfactory results for problems in gas dynamics. However, when applied to the present problem they give merely the same results as the linear theory gives. We shall see that this is because the characteristic velocity contains no term that is linear in the shock amplitude, which implies that the entropy jump is of fourth order in the shock amplitude, rather than third order.

This paper is organized as follows. After setting up the basic equations and determining the shock velocity in Section 2, we derive in Section 3 a modulated transverse cylindrical simple wave theory using the procedure mentioned above. In Section 4, we show how to construct shock wave solutions from the modulated simple wave solutions by using the shock-fitting method, whilst in Section 5 we show how to derive evolution laws using an alternative method, namely that of singular surfaces. The final section is devoted to a discussion of the results obtained in this paper. First, we explain the apparent disagreement between the evolution laws obtained by the two different approximate methods. We go on to discuss the competing decay mechanisms of geometrical spreading and material nonlinearity and conclude that the former is much the more potent. Finally, we show how to recover the evolution laws for plane transverse shock waves from those derived here for cylindrical waves by taking an appropriate limit.

## 2. BASIC EQUATIONS AND THE SHOCK VELOCITY

Let $X_A$, $A = 1, 2, 3$, be the coordinates of a particle in the reference configuration with respect to a rectangular Cartesian coordinate system and $x_i$, $i = 1, 2, 3$, be the coordinates in the current configuration of the same particle with respect to the same coordinate system. A motion is described by the continuous function $x_i = x_i(X_A, t)$ in terms of which the deformation gradient is defined by $F_{iA} = \partial x_i/\partial X_A$.

We consider the propagation of a transverse cylindrical shock wave into an unstrained incompressible isotropic elastic non-conductor $r = (X_1^2+X_2^2)^{1/2} \geqslant r_0$. It can easily be shown with the aid of the jump condition for the conservation of energy that the entropy jump across such a transverse cylindrical shock wave is of fourth order in the shock amplitude. It is well known that the material time derivative of the entropy vanishes away from a shock in an elastic non-conductor. We shall therefore work with the isentropic assumption since this is valid within the order of approximation considered here. The basic equations then simply consist of the equations for the conservation of linear momentum and the constitutive equations. In the material description, considerations of the conservation of linear momentum give the equations of motion

$$\pi_{iA,A} = \rho \dot{v}_i \tag{1}$$

away from a shock in the absence of body forces and yield the jump conditions

$$[\rho U_N v_i + \pi_{iA} N_A] = 0 \tag{2}$$

across a shock. Here $\pi_{iA}$ is the first Piola–Kirchhoff stress tensor and $\rho$, $U_N$ and $N_A$ are in turn the mass density, the propagation speed of the shock and the unit normal to the shock surface in the reference configuration (which we take to be the undisturbed state). The vector $N_A$ is, of course, a radial vector in the $(X_1, X_2)$-plane. For any quantity $f$, the jump across the shock surface is defined by

$$[f] = f^- - f^+,$$

superscripts "+" and "−" signifying evaluation just ahead of the shock and immediately behind the shock, respectively.

For an elastic non-conductor which suffers the single constraint of incompressibility, that is,

$$J \overset{\text{def}}{=} \det \mathbf{F} = 1,$$

the total stress is the sum of a constitutive stress and a reaction stress associated with the constraint:

$$\pi_{iA} = \pi^c_{iA} + p\pi^\alpha_{iA} \tag{3}$$

where

$$\pi^c_{iA} = \rho \frac{\partial \varepsilon}{\partial F_{iA}}, \quad \pi^\alpha_{iA} = JF^{-1}_{Ai}. \tag{4}$$

The function $p(X_A, t)$ is an arbitrary pressure, not directly dependent on $F_{iA}$, which is chosen so that the equations of motion and the boundary conditions are satisfied. The factor $J$ has been allowed to remain in the definition (4b) of the constraint stress, even though it is equal to unity, in order to preserve certain skew-symmetric tensor properties stated in the next paragraph. This is merely for convenience, however, as in either case the arbitrary pressure $p$ multiplying the constraint tensors does not appear in the final equations. The specific internal energy function $\varepsilon$ is independent of the entropy under the isentropic assumption and because the material is both isotropic and incompressible depends on the deformation gradient $F_{iA}$ only through the two invariants

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \tfrac{1}{2}\operatorname{tr}\mathbf{C}^2$$

where $\mathbf{C} \overset{\text{def}}{=} \mathbf{F}^T\mathbf{F}$ is the right Cauchy–Green strain tensor. Thus $\varepsilon = \varepsilon(I_1, I_2)$.

For later use, we define

$$E^c_{iA/B} = \frac{\partial \pi^c_{iA}}{\partial F_{jB}}, \qquad E^\alpha_{iA/B} = \frac{\partial \pi^\alpha_{iA}}{\partial F_{jB}},$$

$$\tilde{E}^c_{iA/BkC} = \frac{\partial^2 \pi^c_{iA}}{\partial F_{jB}\,\partial F_{kC}}, \qquad \tilde{E}^\alpha_{iA/BkC} = \frac{\partial^2 \pi^\alpha_{iA}}{\partial F_{jB}\,\partial F_{kC}},$$

$$\tilde{\tilde{E}}^c_{iA/BkCID} = \frac{\partial^3 \pi^c_{iA}}{\partial F_{jB}\,\partial F_{kC}\,\partial F_{ID}}, \quad \tilde{\tilde{E}}^\alpha_{iA/BkCID} = \frac{\partial^3 \pi^\alpha_{iA}}{\partial F_{jB}\,\partial F_{kC}\,\partial F_{ID}}. \tag{5}$$

With the aid of (4b), we obtain

$$E^\alpha_{iA/B} = J(F^{-1}_{Ai}F^{-1}_{Bj} - F^{-1}_{Aj}F^{-1}_{Bi}). \tag{6}$$

It is seen that $E^\alpha_{iA/B}$ is skew-symmetric with respect to both $(i, j)$ and $(A, B)$. It can easily be shown with the aid of (4b) that $\tilde{E}^\alpha_{iA/BkC}$ and $\tilde{\tilde{E}}^\alpha_{iA/BkCID}$ are also skew-symmetric with respect to the interchange of any pair of lower case, or upper case, suffices. Therefore, their inner products with any symmetric tensor are zero.

Let $R_i$ denote the unit polarization vector which for the purely azimuthal transverse waves under discussion here is orthogonal to the unit wave normal vector $N_A$ and lies in the $(X_1, X_2)$-plane:

$$\delta_{iA} R_i N_A = 0. \tag{7}$$

Thus $R_i$ is a unit azimuthal vector. We may write

$$[v_i] = \phi R_i, \tag{8}$$

where $\phi$ is the shock amplitude. It is convenient to define two other unit vectors $M_A$ and $n_i$ by

$$M_A = \delta_{iA} R_i, \quad n_i = \delta_{iA} N_A, \tag{9}$$

and it can easily be shown that

$$\frac{\partial M_A}{\partial X_B} = -\frac{1}{r} N_A M_B, \qquad \frac{\partial M_A}{\partial X_A} = 0,$$

$$\frac{\partial R_j}{\partial X_B} = -\frac{1}{r} n_j M_B, \qquad \delta_{jB}\frac{\partial R_j}{\partial X_B} = 0,$$

$$\frac{\partial N_A}{\partial X_B} = \frac{1}{r}(\delta_{AB} - N_A N_B), \quad \frac{\partial N_A}{\partial X_A} = \frac{1}{r}, \tag{10}$$

where subscripts range from 1 to 2 only. The shock surface expands radially from the $X_3$-axis so that for all time we have $N_3 = n_3 = R_3 = M_3 = 0$.

We now proceed to determine the shock velocity. By expanding $[\pi^c_{iA}]$ in a Taylor series about the unstrained state $F^+_{iA} = \delta_{iA}$ and using (8), (5) and the compatibility relation

$$[F_{iA}] = -\frac{N_A}{U_N}[v_i], \tag{11}$$

we obtain

$$[\pi^c_{iA}] = -\frac{\phi}{U_N}E^{c+}_{iAjB}R_jN_B + \frac{\phi^2}{2U_N^2}\tilde{E}^{c+}_{iAjBkC}R_jR_kN_BN_C - \frac{\phi^3}{6U_N^3}\tilde{\tilde{E}}^{c+}_{iAjBkClD}R_jR_kR_lN_BN_CN_D + O(\phi^4). \tag{12}$$

Similarly,

$$[\pi^{\alpha}_{iA}] = -\frac{\phi}{U_N}E^{\alpha+}_{iAjB}R_jN_B + \frac{\phi^2}{2U_N^2}\tilde{E}^{\alpha+}_{iAjBkC}R_jR_kN_BN_C - \frac{\phi^3}{6U_N^3}\tilde{\tilde{E}}^{\alpha+}_{iAjBkClD}R_jR_kR_lN_BN_CN_D + O(\phi^4). \tag{13}$$

Therefore, we have

$$[\pi_{iA}] = -\frac{\phi}{U_N}E^{+}_{iAjB}R_jN_B + \frac{\phi^2}{2U_N^2}\tilde{E}^{+}_{iAjBkC}R_jR_kN_BN_C - \frac{\phi^3}{6U_N^3}\tilde{\tilde{E}}^{+}_{iAjBkClD}R_jR_kR_lN_BN_CN_D + [p]\left(\pi^{\alpha+}_{iA} - \frac{\phi}{U_N}E^{\alpha+}_{iAjB}R_jN_B + \frac{\phi^2}{2U_N^2}\tilde{E}^{\alpha+}_{iAjBkC}R_jR_kN_BN_C\right) + \max\{O(\phi^4), O([p]\phi^3)\}, \tag{14}$$

where

$$E^{+}_{iAjB} \overset{\text{def}}{=} E^{c+}_{iAjB} + p^+E^{\alpha+}_{iAjB},$$
$$\tilde{E}^{+}_{iAjBkC} \overset{\text{def}}{=} \tilde{E}^{c+}_{iAjBkC} + p^+\tilde{E}^{\alpha+}_{iAjBkC}. \tag{15}$$

On entering (14) into the jump condition (2) and using the skew-symmetric properties of $E^{\alpha+}_{iAjB}$ and $\tilde{E}^{\alpha+}_{iAjBkC}$, we obtain

$$\rho U_N R_i\phi - \frac{\phi}{U_N}Q^c_{ij}R_j + \frac{\phi^2}{2U_N^2}\tilde{Q}^c_{ijk}R_jR_k - \frac{\phi^3}{6U_N^3}\tilde{\tilde{Q}}^c_{ijkl}R_jR_kR_l + [p]\pi^{\alpha+}_{iA}N_A = \max\{O(\phi^4), O([p]\phi^3)\}, \tag{16}$$

where

$$Q^c_{ij} \overset{\text{def}}{=} E^{c+}_{iAjB}N_AN_B,$$
$$\tilde{Q}^c_{ijk} \overset{\text{def}}{=} \tilde{E}^{c+}_{iAjBkC}N_AN_BN_C,$$
$$\tilde{\tilde{Q}}^c_{ijkl} \overset{\text{def}}{=} \tilde{\tilde{E}}^{c+}_{iAjBkClD}N_AN_BN_CN_D. \tag{17}$$

By evaluating (4b) at the wave front, we see that $\pi^{\alpha+}_{iA} = \delta_{iA}$. Therefore, by (9b),

$$\pi^{\alpha+}_{iA}N_A = n_i. \tag{18}$$

It can be shown (see Fu, 1988) that for the unstrained isotropic material considered here $Q^c_{ij}R_j$ and $\tilde{\tilde{Q}}^c_{ijkl}R_jR_kR_l$ are both aligned with the azimuthal vector $R_i$, whilst $\tilde{Q}^c_{ijk}R_jR_k$ is aligned with the radial vector $n_i$. We may therefore write

$$Q^c_{ij}R_j = \rho\bar{U}^2_N R_i,\quad \tilde{Q}^c_{ijk}R_jR_k = \mu_1 n_i,\quad \tilde{\tilde{Q}}^c_{ijkl}R_jR_kR_l = \mu_2 R_i, \tag{19}$$

where

$$\rho\bar{U}^2_N \overset{\text{def}}{=} Q^c_{ij}R_iR_j,\quad \mu_1 \overset{\text{def}}{=} \tilde{Q}^c_{ijk}n_iR_jR_k,\quad \mu_2 \overset{\text{def}}{=} \tilde{\tilde{Q}}^c_{ijkl}R_iR_jR_kR_l. \tag{20}$$

It is possible to give more explicit expressions (see Fu, 1988, p. 200) for the quantities appearing in (20):

$$\begin{aligned} \rho\bar{U}^2_N &= 2\rho(\varepsilon_1+2\varepsilon_2), \\ \mu_1 &= 4\rho(\varepsilon_2+\varepsilon_{11}+3\varepsilon_{12}+2\varepsilon_{22}), \\ \mu_2 &= 12\rho(\varepsilon_2+\varepsilon_{11}+4\varepsilon_{12}+4\varepsilon_{22}), \end{aligned} \tag{21}$$

where, for example, $\varepsilon_2$ denotes $\partial\varepsilon(I_1, I_2)/\partial I_2$ evaluated in the unstrained state. For future use, we also write down here the following relations which have been shown (see Fu, 1988) to hold for isotropic materials:

$$E^{c*}_{iAjB}R_iR_jN_B = \rho\bar{U}^2_N N_A, \tag{22}$$

$$\tilde{E}^{c*}_{iAjBkC}n_iR_jR_kN_BN_C = \mu_1 N_A, \tag{23}$$

$$\tilde{E}^{c*}_{iAjBkC}R_in_jR_kM_AN_BN_C = 0, \tag{24}$$

$$\tilde{E}^{c*}_{iAjBkC}R_iR_jR_k\delta_{AB}N_C = 0. \tag{25}$$

With the further use of relations (19), we reduce (16) to

$$\left(\rho U^2_N - \rho\bar{U}^2_N - \frac{\mu_2}{6U^2_N}\phi^2\right)\phi R_i + \left(\frac{\mu_1}{2U_N}\phi^2 + U_N[p]\right)n_i = \max\{O(\phi^4), O([p]\phi^3)\}. \tag{26}$$

Since $R_i$ and $n_i$ are orthogonal, relation (26) implies that

$$\rho U^2_N = \rho\bar{U}^2_N + \frac{\mu_2}{6\bar{U}^2_N}\phi^2 + O(\phi^4), \tag{27}$$

$$[p] = -\frac{\mu_1}{2\bar{U}^2_N}\phi^2 + O(\phi^4), \tag{28}$$

so that the order term in (26) may be replaced simply by $O(\phi^4)$.

It can be shown that the jump form of the second law of thermodynamics together with the expression for the entropy jump imposes the condition

$$\mu_2 \geqslant 0, \tag{29}$$

which must be satisfied by a given material if it is to transmit transverse shock waves. This condition is also sufficient to ensure that the propagation speed of a transverse shock is an increasing function of the shock amplitude, as is clear from (27). It is also important to note that eqn (27) contains no term that is linear in the shock amplitude, a fact that was mentioned in the Introduction and is in sharp contrast to the situation for dilatational shock waves [see Fu and Scott, 1989a, eqn (3.6); 1989c, eqn (3.9)]. We see from eqn (28) that the jump suffered by the arbitrary pressure is of only second order in the shock amplitude.

## 3. MODULATED SIMPLE WAVE THEORY

In this section, we establish a modulated simple wave theory for transverse cylindrical motions, which is the basis of the shock-fitting method to be used in the next section. To fix ideas, we consider the motion of an unstrained incompressible isotropic elastic non-conductor initially in a state of rest occupying the region $r \geqslant r_0$ with the purely azimuthal velocity $v_i$ at the boundary prescribed by

$$v_i|_{r=r_0} = \begin{cases} w(t)R_i, & 0 \leqslant t \leqslant T, \\ 0, & \text{otherwise,} \end{cases} \tag{30}$$

where $w(t)$ is a given function of small magnitude. By integrating (30) we see that specifying the azimuthal velocity also specifies the azimuthal displacement on the boundary and so the present problem is equivalent to a displacement boundary value problem. To characterize the wave amplitude, we introduce a dimensionless small parameter defined by

$$\varepsilon = \frac{1}{\bar{U}_N} \max_{0 \leqslant t \leqslant T} |w(t)|. \tag{31}$$

We assume that the duration time $T$ is small compared with the time scale over which geometrical attenuation becomes significant so that (30) represents a small-amplitude, finite-rate pulse of short duration.

On substituting (3) into (1), we obtain

$$E_{iA/B}F_{jB,A} + p_{,A}\pi^{\alpha}_{iA} = \rho\dot{v}_i, \tag{32}$$

where

$$E_{iA/B} = E^{e}_{iA/B} + pE^{\alpha}_{iA/B}. \tag{33}$$

These equations are to be solved, together with the compatibility relations

$$\dot{F}_{jB} = v_{j,B}, \tag{34}$$

subject to the boundary conditions (30).

We may assume that conditions at the boundary $r = r_0$ are carried into the region $r > r_0$ by wavelets subject to geometrical attenuation. If we use $\varphi(r, t) = \alpha$ to denote the wavelet which leaves the boundary at time $\alpha$, we can then take $(X_A, \varphi)$ in place of $(X_A, t)$ as the independent variables and we have the relation

$$\varphi(r_0, \alpha) = \alpha. \tag{35}$$

However, since according to our small-amplitude, finite-rate assumption, the variations of the quantities $F_{iA}$ and $v_i$ with respect to $\varphi$ are two orders of magnitude greater than their variations with respect to $X_A$ for purely transverse waves, we should replace $\varphi$ by the scaled variable $\theta$ defined by

$$\theta = \frac{\varphi(r, t)}{\varepsilon^2}, \tag{36}$$

so that we have

$$F_{iA} = \hat{F}_{iA}(X_A, \theta), \tag{37}$$

$$v_i = \hat{v}_i(X_A, \theta). \tag{38}$$

$$p = \hat{p}(r, \theta). \tag{39}$$

This argument clearly has the same motivation as the method of matched asymptotic expansions (see Nayfeh, 1973, p. 110), which is often used in dealing with boundary layer problems. Here $0 \leqslant x \leqslant T$ can be taken to be the inner region, $x > T$ the outer region. Note that we are only concerned with the inner region.

We look for perturbation solutions of (32) and (34) of the form

$$\begin{aligned} H_{iA} &= \varepsilon H_{iA}^{(1)}(X_A, \theta) + \varepsilon^3 H_{iA}^{(3)}(X_A, \theta) + \cdots, \\ v_i &= \varepsilon v_i^{(1)}(X_A, \theta) + \varepsilon^3 v_i^{(3)}(X_A, \theta) + \cdots, \\ p &= p^+ + \varepsilon p_1(r, \theta) + \varepsilon^2 p_2(r, \theta) + \cdots, \end{aligned} \tag{40}$$

where the displacement gradient is defined by

$$H_{iA} = F_{iA} - \delta_{iA}$$

and assumed to be $O(\varepsilon)$ initially. Because of the purely transverse nature of the waves discussed here, no even powers of $\varepsilon$ are required in the expansions of $H_{iA}$ and $v_i$. The initial arbitrary uniform pressure is denoted by $p^+$. As has already been remarked we shall find that the arbitrary pressure $p$ does not appear in the final equations.

To simplify the analysis, we first contract (3) with $R_i$. It can be shown that $F_{Ai}^{-1} N_A$ is aligned with $n_i$, and by (4b) so is $\pi_{iA}^* N_A$. We therefore have

$$E_{iAjB} R_i F_{jB,A} = \rho \dot{v}_i R_i, \tag{41}$$

where we have used the orthogonality relation (7) and the fact that $p_{,A}$ is proportional to $N_A$.

Expanding $E_{iAjB}$ about the natural state and using (40), we have

$$\begin{aligned} E_{iAjB} = E_{iAjB}^{*} &+ \varepsilon(\tilde{E}_{iAjBkC}^{*} H_{kC}^{(1)} + p_1 E_{iAjB}^{\pi *}) \\ &+ \varepsilon^2(\tfrac{1}{2}\tilde{\tilde{E}}_{iAjBkClD}^{*} H_{kC}^{(1)} H_{lD}^{(1)} + p_2 E_{iAjB}^{\pi *} + p_1 \tilde{E}_{iAjBkC}^{\pi *} H_{kC}^{(1)}) + \cdots. \end{aligned} \tag{42}$$

On substituting (40) and (42) into (34) and (41) and replacing $(\ )_{,A}$ and $(\dot{\ })$ respectively by

$$\frac{\partial}{\partial X_A} + \frac{1}{\varepsilon^2} \varphi_{,r} N_A \frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{1}{\varepsilon^2} \varphi_{,t} \frac{\partial}{\partial \theta},$$

we obtain

$$\left\{\varphi_{,r} N_A \frac{\partial}{\partial \theta} + \varepsilon^2 \frac{\partial}{\partial X_A}\right\} \cdot \{v_i^{(1)}(X_A, \theta) + \varepsilon^2 v_i^{(3)}(X_A, \theta) + \cdots\}$$
$$= \varphi_{,t} \frac{\partial}{\partial \theta} \{H_{iA}^{(1)}(X_A, \theta) + \varepsilon^2 H_{iA}^{(3)}(X_A, \theta) + \cdots\} \tag{43}$$

and

$$\begin{aligned} &\{E_{iAjB}^{*} + \varepsilon(\tilde{E}_{iAjBkC}^{*} H_{kC}^{(1)} + p_1 E_{iAjB}^{\pi *}) \\ &\quad + \varepsilon^2(\tfrac{1}{2}\tilde{\tilde{E}}_{iAjBkClD}^{*} H_{kC}^{(1)} H_{lD}^{(1)} + p_2 E_{iAjB}^{\pi *} + p_1 \tilde{E}_{iAjBkC}^{\pi *} H_{kC}^{(1)}) + \cdots\} \\ &\quad \cdot R_i \left\{\varphi_{,r} N_A \frac{\partial}{\partial \theta} + \varepsilon^2 \frac{\partial}{\partial X_A}\right\} \cdot \{H_{jB}^{(1)}(X_A, \theta) + \varepsilon^2 H_{jB}^{(3)}(X_A, \theta) + \cdots\} \\ &= \rho R_i \varphi_{,t} \frac{\partial}{\partial \theta} \{v_i^{(1)}(X_A, \theta) + \varepsilon^2 v_i^{(3)}(X_A, \theta) + \cdots\}. \end{aligned} \tag{44}$$

As a consequence of the purely transverse nature of the wave motion the term in $\varepsilon$ in (44) may be shown to vanish identically. The leading order and next order terms of (43) and (44) then yield the following four partial differential equations:

$$\varphi_{,t}\frac{\partial H_{jB}^{(1)}}{\partial \theta} = \varphi_{,r}N_B\frac{\partial v_j^{(1)}}{\partial \theta}, \tag{45}$$

$$\varphi_{,t}\frac{\partial H_{jB}^{(3)}}{\partial \theta} = \varphi_{,r}N_B\frac{\partial v_j^{(3)}}{\partial \theta} + \frac{\partial v_j^{(1)}}{\partial X_B}, \tag{46}$$

$$\varphi_{,r}E^{+}_{iAjB}R_iN_A\frac{\partial H_{jB}^{(1)}}{\partial \theta} = \rho\varphi_{,t}R_i\frac{\partial v_i^{(1)}}{\partial \theta}, \tag{47}$$

$$E^{+}_{iAjB}R_i\left(\varphi_{,r}N_A\frac{\partial H_{jB}^{(3)}}{\partial \theta} + \frac{\partial H_{jB}^{(1)}}{\partial X_A}\right) + \left(\tfrac{1}{2}\tilde{\tilde{E}}^{+}_{iAjBkClD}H_{kC}^{(1)}H_{lD}^{(1)} + p_2E^{\alpha +}_{iAjB}\right.$$
$$\left. + p_1\tilde{E}^{\alpha +}_{iAjBkC}H_{kC}^{(1)}\right)\varphi_{,r}R_iN_A\frac{\partial H_{jB}^{(1)}}{\partial \theta} = \rho\varphi_{,t}R_i\frac{\partial v_i^{(3)}}{\partial \theta}. \tag{48}$$

To solve this set of equations, we first substitute for $\partial H_{jB}^{(1)}/\partial\theta$ from (45) into (47) to obtain

$$\left(\rho\left(\frac{\varphi_{,t}}{\varphi_{,r}}\right)^2 - \rho\bar{U}_N^2\right)R_i\frac{\partial v_i^{(1)}}{\partial \theta} = 0, \tag{49}$$

where we have made use of (15a), (17a), (20a), the skew-symmetric property of $E^{\alpha}_{iAjB}$ and the fact that $v_i^{(1)}$ and $\partial v_i^{(1)}/\partial\theta$ are both aligned with $R_i$. Denoting $R_iv_i^{(1)}$ by $a(r,\theta)$, we may write

$$v_i^{(1)} = a(r,\theta)R_i. \tag{50}$$

Since $\partial a/\partial\theta$ should not vanish identically, (49) implies that

$$\frac{\varphi_{,t}}{\varphi_{,r}} = -\bar{U}_N \tag{51}$$

for outgoing wavelets. Integration of (51) subject to the boundary condition (35) then gives

$$\varphi = t - \frac{r - r_0}{\bar{U}_N}. \tag{52}$$

If the motion does not contain any shocks, then both $H_{jB}^{(1)}$ and $v_j^{(1)}$ vanish at $\theta = 0$ and integrating (45) from $\theta = 0$ to $\theta = \theta$ with the use of (50) and (52) yields

$$H_{jB}^{(1)} = -\frac{1}{\bar{U}_N}N_BR_ja. \tag{53}$$

We proceed now to the next order approximation. On substituting for $\partial H_{jB}^{(3)}/\partial\theta$ from (46) into (48) and making use of (50)–(53), we obtain

$$E^{\alpha +}_{iA jB} R_i N_A \frac{\partial(R_j a)}{\partial X_B} + E^{\alpha +}_{iA jB} R_i \frac{\partial(N_B R_j a)}{\partial X_A}$$

$$- \frac{1}{2\bar{U}_N^3} \bar{\bar{E}}^{\alpha +}_{iA jBkCID} R_i N_A R_j N_B R_k N_C R_l N_D a^2 \frac{\partial a}{\partial \theta} = 0, \quad (54)$$

where use has also been made of (19b) and the skew-symmetric properties of $E^{\alpha +}_{iA jB}$, $\bar{E}^{\alpha +}_{iA jBkC}$ and $\bar{\bar{E}}^{\alpha +}_{iA jBkCID}$. With the use of relations (10), (54) becomes

$$2\rho \bar{U}_N^2 \frac{\partial a}{\partial r} - \rho \bar{U}_N^2 \frac{a}{r} - E^{\alpha +}_{iA jB}(R_i n_j N_A M_B + R_i n_j M_A N_B - R_i R_j \delta_{BA}) \frac{a}{r}$$

$$- \frac{1}{2\bar{U}_N^3} \bar{\bar{E}}^{\alpha +}_{iA jBkCID} R_i N_A R_j N_B R_k N_C R_l N_D a^2 \frac{\partial a}{\partial \theta} = 0. \quad (55)$$

It can easily be shown with the aid of (6) that

$$E^{\alpha +}_{iA jB} R_i n_j N_A M_B = -1,$$
$$E^{\alpha +}_{iA jB} R_i n_j M_A N_B = 1,$$
$$E^{\alpha +}_{iA jB} R_i R_j \delta_{AB} = 0, \quad (56)$$

while differentiating (22) with respect to $X_A$ and using (10) gives

$$E^{\alpha +}_{iA jB}(R_i n_j N_A M_B + R_i n_j M_A N_B - R_i R_j \delta_{AB}) = -2\rho \bar{U}_N^2. \quad (57)$$

On inserting (56) and (57) into (55) and noting (15a) and (20c), we arrive at

$$\frac{\partial a}{\partial r} + \frac{a}{2r} - \frac{\beta}{4\bar{U}_N^3} a^2 \frac{\partial a}{\partial \theta} = 0, \quad (58)$$

where

$$\beta \overset{\text{def}}{=} \frac{\mu_2}{\rho \bar{U}_N^2} = 6 \frac{\varepsilon_2 + \varepsilon_{11} + 4\varepsilon_{12} + 4\varepsilon_{22}}{\varepsilon_1 + 2\varepsilon_2} \quad (59)$$

is a dimensionless quantity whose magnitude is a measure of the degree of material nonlinearity. The third term of (58) arises, therefore, from material nonlinearities but the second represents the effect of curvature. Thus eqn (58) expresses a balance between nonlinear effects and geometrical effects.

On defining new variables

$$s = r_0 \log \frac{r}{r_0}, \quad \hat{a}(s, \theta) = \sqrt{\frac{r}{r_0}}\, a(r, \theta)$$

we find that (58) reduces to

$$\frac{\partial \hat{a}}{\partial s} - \frac{\beta}{4\bar{U}_N^3} \hat{a}^2 \frac{\partial \hat{a}}{\partial \theta} = 0 \quad (60)$$

subject to the boundary condition at $s = 0$ (i.e. at $r = r_0$)

$$\varepsilon \hat{a}(0, \theta) = w(\varepsilon^2 \theta) \quad (61)$$

obtained from (30). An equation of the same form as (60) was obtained in a study of plane transverse shear waves [see Lardner (1985), equation preceding (12)]. The solution of (60) is obtained in the standard way by observing that

$$\hat{a}(s,\theta) = \text{constant}$$

along curves governed by

$$\frac{d\theta}{ds} = -\frac{\beta}{4\bar{U}_N^3}\hat{a}^2,$$

so that,

$$\hat{a} = \hat{a}(0,\theta), \quad \theta-\theta_0 = -\frac{\beta}{4\bar{U}_N^3}\hat{a}^2(0,\theta)s, \tag{62}$$

where $\theta = \theta_0$ at $s = 0$. With the use of (30), (61), (36), (40) and (50), we may recast the solution (62) in terms of the original variables:

$$v_i = \sqrt{\frac{r_0}{r}}\,w(\alpha)R_i, \quad t = \alpha + \frac{r-r_0}{\bar{U}_N} - \frac{\beta r_0}{4\bar{U}_N^3}w^2(\alpha)\log\frac{r}{r_0}. \tag{63}$$

Relations (63) constitute the desired modulated transverse cylindrical simple wave solution of equation (58) subject to the boundary condition (30). If $w(\alpha)$ is monotonic equation (63b) can be solved for $\alpha$ in terms of $r$ and $t$, so that (63a) then gives the velocity distribution explicitly in the simple wave region.

We can see from the nonlinear solutions (63) that the wave amplitude has the same form as in the linear theory, but the characteristic variable $\alpha$ is constant along characteristics (63b) determined using the nonlinear theory. Integration of the nonlinear characteristic velocity $dX/dt = (E_{iA/B}N_A N_B R_i R_j/\rho)^{1/2}$ gives the same expression as (63b). Our theory can be shown to give the same results as the nonlinearization technique described by Whitham (1974, Chapter 9).

If instead of propagating into an unstrained quiescent region, the wavelets are preceded by a shock and the shock is advancing into an unstrained quiescent region, then because of the reflection from the shock, the motion behind the shock is not exactly a modulated simple wave. However, substituting (27) and (40) into (11) shows that (53) is satisfied immediately behind the shock. This implies that the reaction of the shock on the motion behind can be disregarded to leading order and a shock can be fitted into the above modulated simple wave solution.

## 4. THE EVOLUTION LAW OF SHOCK WAVES

In this section we proceed to determine the evolution law for transverse cylindrical shock waves using the above modulated simple wave theory and the shock-fitting method. We assume that $R_i v_i|_{r=r_0} = w(t)$ is of the form shown in Fig. 1, so that a shock is initiated at $t = 0$ at the boundary $r = r_0$. It can easily be deduced from (63) that the graph of $v_i R_i$ versus $t$ can be obtained by translating and stretching the graph of $\sqrt{r_0/r}\,w(\alpha)$ to the right by a distance given by

$$\Delta(w(\alpha)) = \frac{r-r_0}{\bar{U}_N} - \frac{\beta r_0}{4\bar{U}_N^3}w^2(\alpha)\log\frac{r}{r_0},$$

and will be multivalued when $\Delta(\varphi_0) \leqslant t \leqslant \Delta(0)$ if $\beta > 0$. [We note from (29) and (59) that the second law of thermodynamics requires $\beta \geqslant 0$ for the transmission of such transverse shocks to be possible.] To obtain a single-valued solution for $R_i v_i$, we fit a shock into the multivalued region. Assume that the shock position is given by $t = S(r)$ as illustrated in

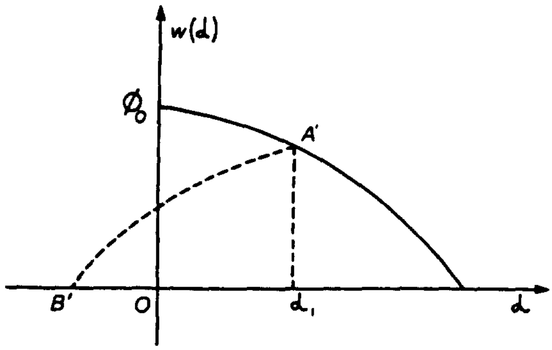


Fig. 1. The initial disturbance $w(t)$ at the boundary $r = r_0$.

Fig. 2, and that when the vertical straight line segment $AB$ is mapped back to its initial position in Fig. 1 it corresponds to the curve $A'B'$. Then we have

$$\phi = [R_i v_i] = \sqrt{\frac{r_0}{r}}\, w(\alpha_1), \tag{64}$$

$$S(r) = \alpha_1 + \frac{r - r_0}{\bar{U}_N} - \frac{\beta r_0}{4\bar{U}_N^3} w^2(\alpha_1) \log \frac{r}{r_0}. \tag{65}$$

In addition, since $S'(r)$ is the reciprocal of the shock speed, we have from (27) that

$$S'(r) = \frac{1}{\bar{U}_N}\left(1 - \frac{\beta}{12\bar{U}_N^2}\phi^2\right). \tag{66}$$

On eliminating $S(r)$ from (64)–(66), we arrive at

$$\left(\frac{1}{w'(\alpha_1)} - \frac{\beta r_0}{2\bar{U}_N^3}\log\frac{r}{r_0}\sqrt{\frac{r}{r_0}}\phi\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{\frac{r}{r_0}}\phi\right) - \frac{\beta}{6\bar{U}_N^3}\frac{r_0}{r}\left(\sqrt{\frac{r}{r_0}}\phi\right)^2 = 0, \tag{67}$$

where $\alpha_1$ can be expressed in terms of the shock amplitude $\phi$ by inverting (64) provided that $w(\alpha)$ is monotonic when $\alpha > 0$. Equation (67) is an ordinary differential equation which can be solved for the shock amplitude $\phi$ once $w(\alpha)$ is given.

As an example, we consider the following loading programme:

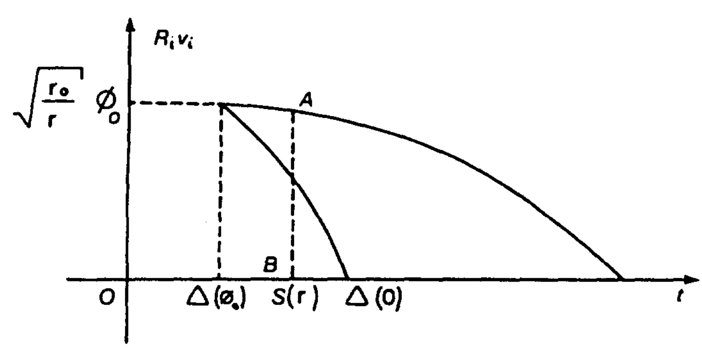


Fig. 2. The subsequent disturbance at interior values $r > r_0$.

$$w(\alpha) = \begin{cases} \phi_0(1-\alpha/T)^m, & 0 \leqslant \alpha \leqslant T, \\ 0, & \text{otherwise}, \end{cases} \tag{68}$$

where $m$ is a positive parameter, $\phi_0 = \phi(0)$, and the duration time $T$ is chosen as

$$T = -\frac{\phi_0 m}{\psi_0}, \tag{69}$$

so that the amplitude of the accompanying second-order discontinuity has the initial value $\psi_0$ which is independent of the parameter $m$. On substituting (68) and (69) into (67), we obtain

$$\frac{1}{\psi_0}\left(\sqrt{\frac{r}{r_0}}\frac{\phi}{\phi_0}\right)^{(1-m)/m}\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{\frac{r}{r_0}}\phi\right) - \frac{\beta r_0}{2\bar{U}_N^3}\sqrt{\frac{r}{r_0}}\left(\log\frac{r}{r_0}\right)\phi\frac{\mathrm{d}}{\mathrm{d}r}\left(\sqrt{\frac{r}{r_0}}\phi\right) - \frac{\beta}{6\bar{U}_N^3}\frac{r_0}{r}\left(\sqrt{\frac{r}{r_0}}\phi\right)^2 = 0. \tag{70}$$

After multiplication by $\sqrt{r/r_0}\,\phi$, (70) may be integrated directly to give

$$y^{1+(1/m)} + \left(1+\frac{1}{m}\right)xy^3 = 1, \tag{71}$$

where the notation

$$x = \frac{\beta}{6\bar{U}_N^3}\phi_0\psi_0 r_0 \log\frac{r_0}{r}, \quad y = \sqrt{\frac{r}{r_0}}\frac{\phi}{\phi_0} \tag{72}$$

has been introduced. Equation (71) is an algebraic equation for the shock amplitude, which can be solved for any positive value of $m$, and shows that the evolution law for a shock wave depends upon $m$ and thus depends upon the detailed boundary conditions. This is in contrast with acceleration waves for which the evolution law depends only on the initial amplitude. An examination of (71) reveals that for decaying shocks we require $x \geqslant 0$ for $r \geqslant r_0$. Thus it is clear from (71) and (72) that the shock decays if the amplitude of the accompanying second-order discontinuity is such that

$$\phi_0\psi_0 \leqslant 0, \tag{73}$$

since $\beta$ defined by (59) is required by the second law of thermodynamics (29) to be non-negative.

## 5. THE SINGULAR SURFACE METHOD

An evolution law for the type of shock wave under discussion can also be derived by using the singular surface method as explained in various different contexts by Fu and Scott (1989a,b,c). Here we give only an outline of the procedure and present the main results. The detailed analysis can be found in Fu (1988) and may be obtained from the first author.

Essential to the singular surface method is the basic compatibility relation

$$\frac{\hat{\partial}}{\partial X_A}[G] = [G_{,A}] + \frac{N_A}{U_N}[\dot{G}], \tag{74}$$

where $G$ is an arbitrary function and $\hat{\partial}/\partial X_A$ is the space derivative following the wave front (see Fu and Scott, 1989c).

Taking the jump of (1) with the use of (74) gives

$$\frac{\hat{\partial}}{\partial X_A}[\pi_{iA}] - \frac{N_A}{U_N}[\dot{\pi}_{iA}] = \rho[\dot{v}_i], \tag{75}$$

while differentiating (1) with respect to time and then taking the jump with the use of (74) yields

$$\frac{\hat{\partial}}{\partial X_A}[\dot{\pi}_{iA}] - \frac{N_A}{U_N}[\ddot{\pi}_{iA}] = \rho[\ddot{v}_i]. \tag{76}$$

The expressions for $\dot{\pi}_{iA}$ and $\ddot{\pi}_{iA}$ can be calculated with the use of the relation (3). Taking the jumps of these expressions and expanding the right-hand sides into Taylor series as we have done in (12), we obtain the approximate expressions for $[\dot{\pi}_{iA}]$ and $[\ddot{\pi}_{iA}]$. On substituting these expressions and (14) and (27) into (75) and (76), we obtain after a great deal of manipulation the following evolution equations:

$$\frac{d\phi}{dr} + \frac{\phi}{2r} - \frac{\beta}{6\bar{U}_N^3}\phi^2\psi = \gamma_1\frac{\phi^3}{r} + \max\{O(\phi^5), O(\phi^4\psi)\}, \tag{77}$$

$$\frac{d\psi}{dr} + \frac{\psi}{2r} + \tfrac{3}{8}\bar{U}_N\frac{\phi}{r^2} - \frac{\beta}{2\bar{U}_N^3}\phi\psi^2$$
$$= \gamma_2\frac{\phi^3}{r^2} + \gamma_3\frac{\phi^2\psi}{r} + \max\{O(\phi^5), O(\phi^4\psi), O(\phi^3\psi^2), O(\phi^2\chi)\}, \tag{78}$$

where

$$\psi \stackrel{\text{def}}{=} R_i[\dot{v}_i], \quad \chi \stackrel{\text{def}}{=} R_i[\ddot{v}_i],$$

denote the second- and third-order discontinuities, respectively. The material constant $\beta$ is defined by (59) but the material constants $\gamma_1$, $\gamma_2$, $\gamma_3$ are not given explicitly since they do not appear in our final results.

The evolution equations (77) and (78) are to be solved simultaneously using perturbation methods, subject to the initial conditions

$$\phi|_{r=r_0} = \phi_0, \quad \text{and} \quad \psi|_{r=r_0} = \psi_0. \tag{79}$$

Two cases must be distinguished. In the first case, the shock amplitude $\phi$ is small but the amplitude $\psi$ of the accompanying second-order discontinuity is of order one (when appropriately scaled). The shock waves covered by this case are of small amplitude and finite rate, and have the same character as those treated in the previous two sections. It can be shown that the leading order solutions of (77) and (78) which are uniformly valid with respect to the distance of travel are given by

$$\phi(r) = \phi_0\sqrt{\frac{r_0}{r}}\left\{1 + \frac{2\beta}{3\bar{U}_N^3}\phi_0\psi_0\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)r_0\log\frac{r_0}{r}\right\}^{-1/4} + \begin{cases} O(\phi_0^3) & \text{if } \chi = O(\phi_0^p), p \geqslant 1, \\ O(\phi_0^2) & \text{if } \chi = O(1). \end{cases} \tag{80}$$

$$\psi(r) = \psi_0\sqrt{\frac{r_0}{r}}\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)\left\{1 + \frac{2\beta}{3\bar{U}_N^3}\phi_0\psi_0\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)r_0\log\frac{r_0}{r}\right\}^{-3/4} + O(\phi_0). \tag{81}$$

In the other case, when $\phi$ and $\psi$ are both small and are of the same order of magnitude, the evolution laws can be shown to be given by

$$\phi(r) = \phi_0\sqrt{\frac{r_0}{r}}\left\{1 + \frac{2\beta}{3\bar{U}_N^3}\phi_0\psi_0\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)r_0\log\frac{r_0}{r}\right\}^{-1/4} + \begin{cases} O(\phi_0^3), & \text{if } \chi = O(\phi_0^p), p \geqslant 3, \\ O(\phi_0^2), & \text{if } \chi = O(\phi_0^2). \end{cases} \tag{82}$$

$$\begin{aligned}\psi(r) = {} & \psi_0\sqrt{\frac{r_0}{r}}\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)\left\{1 + \frac{2\beta}{3\bar{U}_N^3}\phi_0\psi_0\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)r_0\log\frac{r_0}{r}\right\}^{-3/4} \\ & + \frac{3}{8}\sqrt{\frac{r_0}{r}}\frac{\phi_0\bar{U}_N}{r}\left\{1 + \frac{2\beta}{3\bar{U}_N^3}\phi_0\psi_0\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)r_0\log\frac{r_0}{r}\right\}^{-1/4} \\ & + \begin{cases} O(\phi_0^3), & \text{if } \chi = O(\phi_0^p), p \geqslant 3, \\ O(\phi_0^2), & \text{if } \chi = O(\phi_0^2). \end{cases}\end{aligned} \tag{83}$$

Thus we see that the two pairs of evolution equations (80), (81) and (82), (83) form two sets of evolution laws for the shock amplitude and the amplitude of the accompanying second-order discontinuity, the first valid for the case $\psi_0 = O(1)$ and the second for $\psi_0 = O(\phi_0)$. It can easily be seen that (80) and (82) may be written in the single form

$$\phi(r) = \phi_0\sqrt{\frac{r_0}{r}}\left\{1 + \frac{2\beta}{3\bar{U}_N^3}\phi_0\psi_0\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right)r_0\log\frac{r_0}{r}\right\}^{-1/4} + \begin{cases} O(\phi_0^3), & \text{if } \chi = O(\psi_0^2\phi_0^p), p \geqslant 1, \\ O(\phi_0^2), & \text{if } \chi = O(\psi_0^2). \end{cases} \tag{84}$$

Although (83) is different from (81), it reduces to (81) if we take $\psi_0 = O(1)$ since the second term of (83) may then be neglected compared with the first. Therefore, eqns (84) and (83) can be taken to be the universal evolution laws for the shock amplitude and the amplitude of the accompanying second-order discontinuity.

In the case $\psi_0 = O(1)$ the evolution law (80) may be written

$$y = \{1 + 4x\}^{-1/4} \tag{85}$$

in terms of the notation (72).

## 6. DISCUSSION

On comparing the evolution laws (71) with (84), which are derived respectively by using the shock-fitting method and the singular surface method, we see that they do not agree exactly for any value of $m$. Agreement is not to be expected in the case $\psi_0 = O(\phi_0)$ for then the small-amplitude, finite-rate assumption made in deriving (71) is violated. Neither is there exact agreement, however, between (71) and (85) in the case $\psi_0 = O(1)$ for which such agreement might have been expected. In order to examine the discrepancies we expand both (71) and (85) for small $x$:

$$\phi = \phi_0 \exp(-x/\delta)\left\{1 - x + \frac{6m-1}{2m}x^2 - \frac{(4m-1)(9m-1)}{3m^2}x^3 + \cdots\right\}, \tag{86}$$

$$\phi = \phi_0 \exp(-x/\delta)\{1 - x + \tfrac{5}{2}x^2 - \tfrac{15}{2}x^3 + \cdots\}, \tag{87}$$

respectively, and for large $x$:

$$\phi \sim \phi_0 \exp(-x/\delta)\{(1+1/m)x\}^{-1/3}, \tag{88}$$

$$\phi \sim \phi_0 \exp(-x/\delta)(4x)^{-1/4}, \tag{89}$$

respectively. The dimensionless quantity $\delta$ defined by

$$\delta \overset{\text{def}}{=} -\frac{\beta}{3\bar{U}_N^3}\phi_0\psi_0 r_0$$

is small and positive. For small $x$ the expansions agree as far as the linear term; they agree as far as the quadratic term for the linear loading programme (68) in which $m = 1$. For large $x$ both asymptotic expansions predict a shock amplitude $\phi$ that is exponentially small compared with its initial magnitude. Hence for all $x$, the difference between the solutions given by (71) and (85) is at most of order $\phi_0^3$ if $m \neq 1$ and $\phi_0^4$ if $m = 1$. Therefore we may conclude that these solutions do, in fact, agree with each other, at least to leading order.

We now discuss the effects of material nonlinearity on the evolution of transverse cylindrical shock waves. We have already remarked that the material constant $\beta$ defined by (59) is a measure of the degree of material nonlinearity present and we now observe that it enters into the asymptotic expressions for the shock amplitude solely through the variable $x$ defined by (72a) which increases only logarithmically with the radial distance $r$. Thus we see that both (71) and (84) display an important fact, namely, that the effects of nonlinearity on the evolution of transverse cylindrical shock waves are cumulative and are most pronounced when $x = O(1)$, that is, when $r$ satisfies the order relation

$$r/r_0 = O(\exp(2/\delta)).$$

However, at distances of this order the shock amplitude has already been attenuated by geometrical spreading to order $\exp(-1/\delta)$. Therefore, by the time nonlinear effects have become significant, geometrical spreading has already made the shock amplitude exponentially small. Alternatively, we may say that over distances comparable with the initial radius $r_0$, the variable $x$ which measures the effect of nonlinearity remains close to zero. Arguing from either standpoint, we may conclude that the linear evolution laws

$$\phi = \phi_0\sqrt{\frac{r_0}{r}}, \quad \psi = \psi_0\sqrt{\frac{r_0}{r}}\left(1 - \frac{3\bar{U}_N}{8r_0}\frac{\phi_0}{\psi_0}\right) + \frac{3}{8}\sqrt{\frac{r_0}{r}}\frac{\phi_0\bar{U}_N}{r}, \tag{90}$$

obtained by taking $\beta = 0$ in (84) and in (83), give satisfactory results for the propagation of transverse cylindrical shock waves in unstrained isotropic elastic non-conductors. This is in sharp contrast with plane transverse shock waves in similar materials for which the linear theory predicts constant values for the amplitudes $\phi$ and $\psi$, whilst the nonlinear theory predicts amplitudes that decay with the distance of travel [see Fu and Scott (1989b), eqns (4.52) and (4.53)]. We must conclude that geometrical spreading is a more potent decay mechanism than is material nonlinearity.

To complete the picture we describe how to obtain the above-mentioned plane wave solutions from the cylindrical wave solutions presented here. If $r_0 \to \infty$ and $r \to \infty$ in such a way that $X \equiv r - r_0$ remains finite then we have

$$\sqrt{\frac{r_0}{r}} \approx 1 - \frac{X}{2r_0}, \quad r_0 \log \frac{r_0}{r} \approx -X.$$

On inserting these approximations into (84) and (83) and taking the limits $r_0 \to \infty$, $r \to \infty$ we find

$$\phi = \phi_0 \left\{ 1 - \frac{2\beta}{3\bar{U}_N^3} \phi_0 \psi_0 X \right\}^{-1/4}, \quad \psi = \psi_0 \left\{ 1 - \frac{2\beta}{3\bar{U}_N^3} \phi_0 \psi_0 X \right\}^{-3/4}, \tag{91}$$

which are the same as the equations of Fu and Scott [1989b, eqns (4.52) and (4.53)] allowing for differences in notation. By putting $\beta = 0$ in (91) we obtain the constant linear approximations $\phi = \phi_0$ and $\psi = \psi_0$ for plane waves, whilst the linear approximations for cylindrical waves (90) are not constant because of geometrical spreading.

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